

GEOMETRY OF THE FACES FOR SEPARABLE STATES ARISING FROM GENERALIZED CHOI MAPS

KIL-CHAN HA AND SEUNG-HYEOK KYE

ABSTRACT. We exhibit examples of separable states which are on the boundary of the convex cone generated by all separable states but in the interior of the convex cone generated by all PPT states. We also analyze the geometric structures of the smallest face generated by those examples. As a byproduct, we obtain a large class of entangled states with positive partial transposes.

1. INTRODUCTION

The notion of entanglement is now one of the key research topics of quantum physics and considered as the main resources for quantum information and quantum computation theory. One of the most basic research topics in the theory of entanglement is how to distinguish entanglement from separable states. We recall that a state on the tensor product $M_m \otimes M_n$ is said to be separable if it is the convex combination of product states, where M_n denotes the $*$ -algebra of all $n \times n$ matrices over the complex field. If we identify a state on $M_m \otimes M_n$ as a density matrix in $M_m \otimes M_n$, then a density matrix is separable if and only if it is the convex sum of rank one projections onto product vectors. A density matrix in $M_m \otimes M_n$ is entangled if it is not separable. It was observed by Choi [6] and Peres [21] that the partial transpose of a separable density matrix is again positive semi-definite. This PPT criterion gives us a very simple necessary condition for separability among various criteria for separability. See [3]. The notion of PPT states turns out to be very important in itself in relation with bound entanglement. See [16].

In order to distinguish entanglement from separability, it is important to understand the boundary structures of the convex cone \mathbb{V}_1 generated by all separable states. We note that the boundary $\partial\mathbb{V}_1$ of the cone \mathbb{V}_1 consists of maximal faces. We also note that the PPT criterion tells us that the cone \mathbb{V}_1 is contained in the convex cone \mathbb{T} generated by all PPT states. Therefore, we have the two cases for a maximal face F of \mathbb{V}_1 :

- F is contained in the boundary $\partial\mathbb{T}$ of the cone \mathbb{T} ,
- The interior $\text{int } F$ of F is contained in $\text{int } \mathbb{T}$.

Although the first case is easy to characterize using the boundary structures [11] of the cone \mathbb{T} , there is nothing to be known for the second case. Even, there is no known explicit examples of separable states on the boundary $\partial\mathbb{V}_1$ of \mathbb{V}_1 , but in the interior of the cone \mathbb{T} , to the best knowledge of the authors. The main purpose of this note is to exhibit such examples in the case

1991 *Mathematics Subject Classification.* 81P15, 15A30, 46L05.

Key words and phrases. separable, entanglement, positive partial transpose, face.

KCH is partially supported by NRFK 2011-0006561. SHK is partially supported by NRFK 2012-0000939.

of $M_3 \otimes M_3$, and investigate the geometric structures of the maximal faces determined by those examples. As a byproduct of those examples, we naturally get examples of entangled states with positive partial transposes whose ranges are full spaces.

Note that every point x of a convex set C determines the smallest face containing the point. This is the unique face of C in which x is an interior point. In the next section, we provide a general method how to understand the face of the cone \mathbb{V}_1 determined by a given separable state. After we give examples of separable states in $\partial\mathbb{V}_1 \cap \text{int}\mathbb{T}$ in Section 3, we analyze the faces determined by them in the last two sections. Throughout this note, a vector z will be considered as a column vector, and we denote by \bar{z} and z^* the complex conjugate and the Hermitian conjugate of z , respectively. So zz^* is the one dimensional projection onto the vector z with this notation, whenever z is a unit vector.

2. FACES FOR SEPARABLE STATES

The convex cone \mathbb{T} consists of positive semi-definite matrices $A = \sum e_{ij} \otimes x_{ij}$ in $M_m \otimes M_n$ whose partial transpose $A^\Gamma = \sum e_{ij} \otimes x_{ji}$ is also positive semi-definite, where $\{e_{ij}\}$ is the usual matrix units in M_m . We recall that a convex subset F of a convex set C is said to be a face of C if the following property holds:

$$x, y \in C, (1-t)x + ty \in F \text{ for some } t \in (0,1) \implies x, y \in F.$$

In short, if an interior point of a line segment lies in the face F then the whole line segment should lie in F . The faces of the convex cone \mathbb{T} are determined [11] by pairs (D, E) of subspaces of $\mathbb{C}^m \otimes \mathbb{C}^n$. More precisely, every face of \mathbb{T} is of the form

$$\tau(D, E) := \{A \in \mathbb{T} : \mathcal{R}A \subset D, \mathcal{R}A^\Gamma \subset E\},$$

where $\mathcal{R}A$ denotes the range space of $A \in M_m \otimes M_n$. We also note that the interior of the face $\tau(D, E)$ is given by

$$(1) \quad \text{int } \tau(D, E) = \{A \in \mathbb{T} : \mathcal{R}A = D, \mathcal{R}A^\Gamma = E\}.$$

We recall that a point x of a convex set C is an interior point of C if the line segment from any point of C to x can be extended within C . A point of C which is not an interior point is said to be a boundary point. We denote by $\text{int } C$ the set of all interior point of C , which is nothing but the relative topological interior of C with respect to the affine manifold generated by C . The relation (1) tells us that a separable state $A \in \mathbb{V}_1$ lies in the interior of the convex cone \mathbb{T} if and only if both $\mathcal{R}A$ and $\mathcal{R}A^\Gamma$ are the full space $\mathbb{C}^m \otimes \mathbb{C}^n$. We also note that if $A \in \mathbb{T}$ then $\tau(\mathcal{R}A, \mathcal{R}A^\Gamma)$ is the smallest face of \mathbb{T} determined by A , which will be denoted by $\mathbb{T}[A]$. That is, we define

$$\mathbb{T}[A] := \tau(\mathcal{R}A, \mathcal{R}A^\Gamma).$$

For a separable state $A \in \mathbb{V}_1$, we denote by $\mathbb{V}_1[A]$ the smallest face of \mathbb{V}_1 determined by A , and define the following two sets of product vectors:

$$\begin{aligned} P[A] &= \{z = x \otimes y : zz^* \in \mathbb{V}_1[A]\}, \\ Q[A] &= \{z = x \otimes y : x \otimes y \in \mathcal{R}A, \bar{x} \otimes y \in \mathcal{R}A^\Gamma\}. \end{aligned}$$

We note that $P[A]$ represents the set of all extreme rays of the face $\mathbb{V}_1[A]$. For a set P of product vectors, we denote by \overline{P} the set of partial conjugates $\bar{x} \otimes y$ of $x \otimes y \in P$. If two separable states $A, B \in \mathbb{V}_1$ lie in the interior of the same face $\tau(D, E)$ of \mathbb{T} , then $Q[A]$ coincides with $Q[B]$. But, $P[A]$ and $P[B]$ may be different, even though they are in the interior of a common face of \mathbb{T} .

Theorem 2.1. *Proposition For every separable state $A \in \mathbb{V}_1$, we have the following:*

- (i) $P[A] \subset Q[A]$.
- (ii) $\text{span } P[A] = \text{span } Q[A] = \mathcal{R}A$.
- (iii) $\text{span } \overline{P[A]} = \text{span } Q[A^\Gamma] = \mathcal{R}A^\Gamma$.

Proof. Suppose that A is expressed by

$$(2) \quad A = \sum_{\iota} (x_{\iota} \otimes y_{\iota})(x_{\iota} \otimes y_{\iota})^*,$$

for $x_{\iota} \otimes y_{\iota} \in \mathbb{C}^m \otimes \mathbb{C}^n$. Then $A^\Gamma = \sum_{\iota} (\bar{x}_{\iota} \otimes y_{\iota})(\bar{x}_{\iota} \otimes y_{\iota})^*$, and we have

$$(3) \quad \mathcal{R}A = \text{span} \{x_{\iota} \otimes y_{\iota}\}, \quad \mathcal{R}A^\Gamma = \text{span} \{\bar{x}_{\iota} \otimes y_{\iota}\},$$

by [17]. Let $z = x \otimes y$ be a product vector in $P[A]$, and so $zz^* \in \mathbb{V}_1[A]$. Since A is an interior point of $\mathbb{V}_1[A]$, we see that A can be written as the sum of zz^* and a separable state $B = \sum z_{\iota} z_{\iota}^* \in \mathbb{V}_1[A]$ with product vectors z_{ι} . Therefore, we see that $x \otimes y \in \mathcal{R}A$, and $\bar{x} \otimes y \in \mathcal{R}A^\Gamma$. This completes the proof of (i).

Since $x \otimes y \in Q[A]$ implies $x \otimes y \in \mathcal{R}A$ by definition, it is clear that $\text{span } Q[A] \subset \mathcal{R}A$. On the other hand, the expression (2) tells us that $(x_{\iota} \otimes y_{\iota})(x_{\iota} \otimes y_{\iota})^*$ belongs to the face $\mathbb{V}_1[A]$, and so $x_{\iota} \otimes y_{\iota} \in P[A]$. By the relation (3), we have $\mathcal{R}A \subset \text{span } P[A]$. Similarly, we also have $\mathcal{R}A^\Gamma \subset \text{span } \overline{P[A]}$. \square

Now, we proceed to compare the face $\mathbb{V}_1[A]$ of the cone \mathbb{V}_1 and the face $\mathbb{T}[A]$ of the cone \mathbb{T} determined by $A \in \mathbb{V}_1 \subset \mathbb{T}$. Since A is an interior point of both of $\mathbb{V}_1[A]$ and $\mathbb{T}[A]$, we see that $\mathbb{V}_1[A]$ lies inside of $\mathbb{T}[A]$. More precisely, we have

$$\text{int } \mathbb{V}_1[A] \subset \text{int } \mathbb{T}[A], \quad \mathbb{V}_1[A] \subset \mathbb{V}_1 \cap \mathbb{T}[A].$$

Since both convex set $\mathbb{V}_1[A]$ and $\mathbb{V}_1 \cap \mathbb{T}[A]$ are faces of the convex cone \mathbb{V}_1 , we have the following two possibilities:

- $\mathbb{V}_1[A] = \mathbb{V}_1 \cap \mathbb{T}[A]$,
- $\mathbb{V}_1[A]$ is a proper face of $\mathbb{V}_1 \cap \mathbb{T}[A]$.

If the first case occurs, then we see that $\mathbb{V}_1[A]$ is induced by the face $\mathbb{T}[A]$ of the cone \mathbb{T} in the sense of [5]. If both $\mathcal{R}A$ and $\mathcal{R}A^\Gamma$ are the full space $\mathbb{C}^m \otimes \mathbb{C}^n$ then the second case gives rise to the boundary of the cone \mathbb{V}_1 which lies in the interior of the cone \mathbb{T} .

Theorem 2.2. *Theorem For a separable state $A \in \mathbb{V}_1$, the following are equivalent:*

- (i) $\mathbb{V}_1[A] = \mathbb{V}_1 \cap \mathbb{T}[A]$
- (ii) A is an interior point of $\mathbb{V}_1 \cap \mathbb{T}[A]$.
- (iii) $P[A] = Q[A]$.

(iv) For every $z \in Q[A]$ there exist product vectors $z_i \in Q(A)$ such that $A = zz^* + \sum_i z_i z_i^*$.

Proof. The equivalence (i) \iff (ii) follows from the definition of $\mathbb{V}_1[A]$. We suppose that the condition (i) holds. Then we have

$$x \otimes y \in P[A] \iff (x \otimes y)(x \otimes y)^* \in \mathbb{T}[A] \iff x \otimes y \in \mathcal{R}A, \bar{x} \otimes y \in \mathcal{R}A^\Gamma,$$

since $\mathbb{T}[A] = \tau(\mathcal{R}A, \mathcal{R}A^\Gamma)$. This proves the direction (i) \implies (iii). To prove the direction (iii) \implies (i), suppose that $P[A] = Q[A]$. It suffices to show that $\mathbb{V}_1[A] \supset \mathbb{V}_1 \cap \tau(\mathcal{R}A, \mathcal{R}A^\Gamma)$. To do this, take arbitrary $B = \sum z_i z_i^* \in \tau(\mathcal{R}A, \mathcal{R}A^\Gamma)$ with $z_i = x_i \otimes y_i$. Then we see that $x_i \otimes y_i \in \mathcal{R}A$ and $\bar{x}_i \otimes y_i \in \mathcal{R}A^\Gamma$, which implies that $z_i \in Q[A] = P[A]$. This means that $z_i z_i^* \in \mathbb{V}_1[A]$, and so we have $B \in \mathbb{V}_1[A]$.

To complete the proof, it remains to show (ii) \iff (iv). To do this, we first note that the set of all extreme rays of the convex cone $\mathbb{V}_1 \cap \mathbb{T}[A]$ consists of zz^* with $z \in Q[A]$, since $zz^* \in \mathbb{T}[A]$ if and only if $z \in Q[A]$. The statement (iv) says that the line segment from zz^* to A can be extended within $\mathbb{V}_1 \cap \mathbb{T}[A]$. Since every element in $\mathbb{V}_1 \cap \mathbb{T}[A]$ is the convex combination of zz^* with $z \in Q[A]$, we see that this is equivalent to say that A is an interior point of $\mathbb{V}_1 \cap \mathbb{T}[A]$. \square

When we consider a specific example of a separable state A , it is not so easy to determine the set $P[A]$ which characterize the extremal rays of the face $\mathbb{V}_1[A]$. We note that it is relatively easy to determine the set $Q[A]$, and so we may determine whether the condition (iv) of Theorem 2.2 holds or not. If the condition (iv) holds, then we may figure out the face $\mathbb{V}_1[A]$ in two ways: It is induced by a face of the bigger cone \mathbb{T} by (i) of Theorem 2.2; we know all extremal rays of the face $\mathbb{V}_1[A]$ by (iii) of Theorem 2.2.

Another way to determine the set $P[A]$ is to use the duality [10, 15] between the convex cone \mathbb{V}_1 and the cone \mathbb{P}_1 consisting of all positive linear maps from M_m into M_n . For $A = \sum_{i,j=1}^m e_{ij} \otimes a_{ij} \in M_m \otimes M_n$ and a linear map ϕ from M_m into M_n , we define the bilinear pairing by

$$\langle A, \phi \rangle = \sum_{i,j=1}^m \text{Tr}(\phi(e_{ij})a_{ij}^t) = \text{Tr}(C_\phi A^t),$$

where $C_\phi = \sum_{i,j=1}^m e_{ij} \otimes \phi(e_{ij})$ is the Choi matrix of the linear map ϕ . We note that every positive linear map ϕ gives rise to the face

$$\phi' = \{A \in \mathbb{V}_1 : \langle A, \phi \rangle = 0\}$$

of \mathbb{V}_1 . We call ϕ' the dual face of the map ϕ with respect to dual pair $(\mathbb{V}_1, \mathbb{P}_1)$. A face is said to be exposed if it is a dual face. It is not known that if every face of the cone \mathbb{V}_1 is exposed, even though it is known [11] that every face of the cone \mathbb{T} is exposed. We refer to [19] for general aspects of duality.

For a product vector $z = x \otimes y \in \mathbb{C}^m \otimes \mathbb{C}^n$, we have

$$(4) \quad \langle zz^*, \phi \rangle = \langle xx^* \otimes yy^*, \phi \rangle = \text{Tr}(\phi(xx^*)\bar{y}y^*) = (\phi(xx^*)\bar{y}|\bar{y}),$$

where $(\cdot | \cdot)$ denotes the inner product in \mathbb{C}^n which is linear in the first variable and conjugate linear in the second variable. Therefore, we have $z = x \otimes y \in P[A]$ if and only if $\langle zz^*, \phi \rangle = 0$

if and only $\phi(xx^*) \in M_n$ is singular and \bar{y} is a kernel vector of $\phi(xx^*)$. In this way, we can determine the set $P[A]$. See [18] for related topics.

3. EXAMPLES

Recall the generalized Choi map $\Phi[\alpha, \beta, \gamma]$ between M_3 defined by

$$\Phi[\alpha, \beta, \gamma](X) = \begin{pmatrix} \alpha x_{11} + \beta x_{22} + \gamma x_{33} & -x_{12} & -x_{13} \\ -x_{21} & \gamma x_{11} + \alpha x_{22} + \beta x_{33} & -x_{23} \\ -x_{31} & -x_{32} & \beta x_{11} + \gamma x_{22} + \alpha x_{33} \end{pmatrix},$$

for $X \in M_3$ and nonnegative real numbers α, β and γ , as was introduced in [4]. It is a positive linear map if and only if the condition

$$\alpha + \beta + \gamma \geq 2, \quad 0 \leq \alpha \leq 1 \implies \beta\gamma \geq (1 - \alpha)^2$$

holds. The Choi matrix $C_\Phi[\alpha, \beta, \gamma]$ of the map $\Phi[\alpha, \beta, \gamma]$ is given by

$$C_\Phi[\alpha, \beta, \gamma] = \begin{pmatrix} \alpha & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & \gamma & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \alpha \end{pmatrix} \in M_3 \otimes M_3.$$

The most interesting cases occur when the following condition

$$(5) \quad 0 \leq \alpha < 1, \quad \alpha + \beta + \gamma = 2, \quad \beta\gamma = (1 - \alpha)^2$$

holds. Motivated by the parametrization in [9] for those cases, the authors [13] have shown that $\Phi[\alpha, \beta, \gamma]$ is an indecomposable exposed positive linear map under the conditions (5). To do this, they considered the another parametrization given by

$$\Phi(t) = \Phi \left[\frac{(1-t)^2}{1-t+t^2}, \frac{t^2}{1-t+t^2}, \frac{1}{1-t+t^2} \right], \quad 0 < t < \infty.$$

See also [8] and [12]. Note that $\Phi(1) = \Phi[0, 1, 1]$ is completely copositive, and both $\Phi(0) = \Phi[1, 0, 1]$ and $\Phi(\infty) = \Phi[1, 1, 0]$ are indecomposable extremal positive maps [7] which are not exposed. See the picture in Section 5 of [19].

We note that the matrix $C_\Phi[\alpha, \beta, \gamma]$ is of PPT if and only if $\alpha \geq 2$ and $\beta\gamma \geq 1$. It turns out [20] that it is in fact separable if $\alpha = 2$ and $\beta\gamma = 1$. Therefore, we see that $C_\Phi[\alpha, \beta, \gamma]$ is of

PPT if and only if it is separable. Our strategy is to consider the following matrix

$$(6) \quad A[a, b, c] = \begin{pmatrix} a & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & a \end{pmatrix} \in M_3 \otimes M_3$$

for nonnegative real numbers a, b and c , and seek the condition for separability using the indecomposable exposed positive linear map $\Phi(t)$. Note that

$$\langle A[a, b, c], \Phi[\alpha, \beta, \gamma] \rangle = \text{Tr}(A[a, b, c]C_\Phi[\alpha, \beta, \gamma]^t) = 3(a\alpha + b\beta + c\gamma - 2).$$

First of all, it is easy to see that A is of PPT if and only if

$$(7) \quad a \geq 1, \quad bc \geq 1,$$

since we have

$$A[a, b, c]^\Gamma = \begin{pmatrix} a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & c & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a \end{pmatrix}.$$

A PPT state A is said to be with type (p, q) if $\dim \mathcal{R}A = p$ and $\dim \mathcal{R}A^\Gamma = q$. We note that $A[a, b, c]$ is of PPT with type

- (7, 6) if $a = 1$ and $bc = 1$,
- (9, 6) if $a > 1$ and $bc = 1$,
- (7, 9) if $a = 1$ and $bc > 1$,
- (9, 9) if $a > 1$ and $bc > 1$.

To get a necessary condition for separability of $A[a, b, c]$, we suppose that $A[a, b, c]$ is separable. Then we have $\langle A[a, b, c], \Phi(t) \rangle \geq 0$ for each $t > 0$. In other words, we have the condition

$$t > 0 \implies \frac{a(1-t)^2}{1-t+t^2} + \frac{bt^2}{1-t+t^2} + \frac{c}{1-t+t^2} \geq 2.$$

This condition holds if and only if

$$t > 0 \implies a(1-t)^2 + bt^2 + c \geq 2(1-t+t^2)$$

if and only if

$$t > 0 \implies (a+b-2)t^2 + 2(1-a)t + (a+c-2) \geq 0$$

if and only if

$$(8) \quad a + b - 2 \geq 0, \quad (b + a - 2)(c + a - 2) \geq (1 - a)^2.$$

Note that (8) implies $a + c - 2 \geq 0$, which may replace $a + b - 2 \geq 0$ in (8). Therefore, we see that if $A[a, b, c]$ is separable then the both conditions (7) and (8) hold. We proceed to show that these are sufficient for separability of $A[a, b, c]$. We note that the condition (8) implies (7) strictly when $1 \leq a < 2$, and two conditions (7) and (8) coincide when $a = 2$.

We denote by C the convex subset consisting of $(a, b, c) \in \mathbb{R}^3$ satisfying the conditions (7) and (8). We see that the following three points

$$(1, 1, 1), \quad (a, y, z), \quad \left(2, \frac{y-1}{a-1} + 1, \frac{z-1}{a-1} + 1\right)$$

are on a single line for $1 < a < 2$, and if the triplet $(a, b, c) = (a, y, z)$ satisfies the equality in the second inequality of (8) then we have

$$\left(\frac{y-1}{a-1} + 1\right) \left(\frac{z-1}{a-1} + 1\right) = 1.$$

Therefore, if we put $b = \frac{y-1}{a-1} + 1$ and $c = \frac{z-1}{a-1} + 1$ then the point $(2, b, c)$ lies on the boundary of the convex body determined by (7) and (8). This means that every boundary point of the convex body

$$C_1 = \{(a, y, z) : 1 < a < 2, a, y, z \text{ satisfy (8)}\}$$

lies on the line segment between the point $(1, 1, 1)$ and a point $(2, b, \frac{1}{b})$ for $b \neq 1$, which are extreme points of the convex body C . In conclusion, the boundary of the convex body C consists of the following:

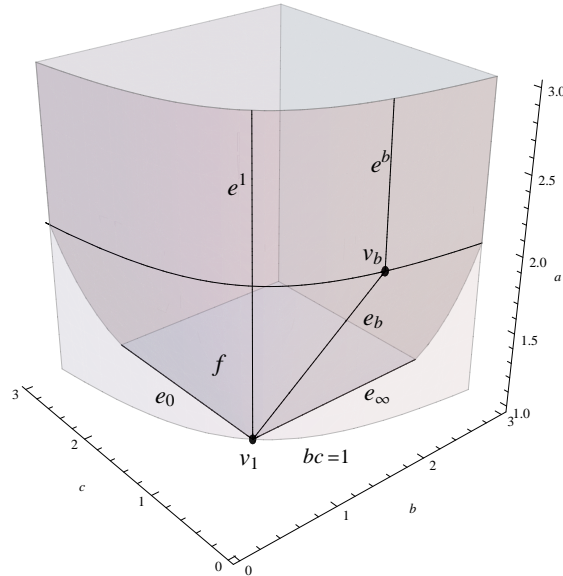


FIGURE 1. The edge e_∞ can be regarded as the limit of e_b as $b \rightarrow \infty$, and e_0 can be regarded as the limit of e_b as $b \rightarrow 0$.

- $v_1 = (1, 1, 1)$

- $v_b = (2, b, \frac{1}{b})$ for $b > 0$ with $b \neq 1$
- $e^1 = \{(a, 1, 1) : a \geq 1\}$
- $e^b = \{(a, b, \frac{1}{b}) : a \geq 2\}$ for $b > 0$ with $b \neq 1$
- e_b = the line segment between v_1 and v_b for $b > 0$ with $b \neq 1$
- $e_0 = \{(1, 1, c) : c \geq 1\}$
- $e_\infty = \{(1, b, 1) : b \geq 1\}$
- $f = \{(1, b, c) : b \geq 1, c \geq 1\}$

Any interior points of edges $e^1, e^b, e_b, e_0, e_\infty$ and two-dimensional face f will be denoted by the same symbols. For example, any point $(a, b, \frac{1}{b})$ with $a > 2$ will be denoted by just e^b . This does not make any confusion, because any two interior points of the ‘edge’ e_b determine the same face as that determined by the edge e_b itself.

In order to prove that the condition (8) together with (7) implies the separability of $A[a, b, c]$, it suffices to show that $A[v_b]$ are separable for each $b > 0$. We first consider the case of $b \neq 1$. To do this, we define product vectors

$$\begin{aligned}
 (9) \quad z_1(\omega) &= (0, 1, \sqrt{b}\omega)^t \otimes (0, \sqrt{b}, \bar{\omega})^t = (0, 0, 0; 0, \sqrt{b}, \bar{\omega}; 0, b\omega, \sqrt{b})^t, \\
 z_2(\omega) &= (\sqrt{b}\omega, 0, 1)^t \otimes (\bar{\omega}, 0, \sqrt{b})^t = (\sqrt{b}, 0, b\omega; 0, 0, 0; \bar{\omega}, 0, \sqrt{b})^t, \\
 z_3(\omega) &= (1, \sqrt{b}\omega, 0)^t \otimes (\sqrt{b}, \bar{\omega}, 0)^t = (\sqrt{b}, \bar{\omega}, 0; b\omega, \sqrt{b}, 0; 0, 0, 0)^t
 \end{aligned}$$

in $\mathbb{C}^3 \otimes \mathbb{C}^3$, for $\omega \in \mathbb{C}$ with $|\omega| = 1$. Then it is straightforward to see that

$$A[v_b] = \frac{1}{3b} \sum_{i=1}^3 \sum_{\omega \in \Omega} z_i(\omega) z_i(\omega)^*,$$

where $\Omega = \{1, e^{\frac{2}{3}\pi i}, e^{-\frac{2}{3}\pi i}\}$ is the third roots of unity. If we put

$$(10) \quad z_4(\omega, \eta) = (1, \bar{\omega}, \bar{\eta}) \otimes (1, \omega, \eta) = (1, \omega, \eta; \bar{\omega}, 1, \bar{\omega}\eta; \bar{\eta}, \omega\bar{\eta}, 1)$$

for $(\omega, \eta) \in \Omega \times \Omega$, then we also have

$$(11) \quad A[v_1] = \frac{1}{9} \sum_{(\omega, \eta) \in \Omega \times \Omega} z_4(\omega, \eta) z_4(\omega, \eta)^*.$$

Therefore, we see that $A[v_1]$ and $A[v_b]$ are separable.

Theorem 3.1. *Theorem The state $A[a, b, c]$ is of PPT if and only if the condition (7) holds, and separable if and only if both conditions (7) and (8) hold.*

Now, we can characterize a large class of entangled states of the form (6) with positive partial transpose.

Theorem 3.2. *Corollary The state $A[a, b, c]$ is PPTES if and only if the condition*

$$1 \leq a < 2, \quad bc \geq 1, \quad (b + a - 2)(c + a - 2) < (1 - a)^2$$

holds.

The state $A[1, b, \frac{1}{b}]$ is nothing but the PPTES considered in [22] in the early eighties, which has been reconstructed in [14] systematically using the indecomposable positive linear maps.

4. MAXIMAL FACES

In this section, we determine the faces $\mathbb{V}_1[A]$ for separable state A which comes from the boundaries of the convex body C . We first determine the sets $Q[A[v_1]]$ and $Q[A[v_b]]$. We note that the kernel of $A[v_1]$ is spanned by

$$(1, 0, 0; 0, -1, 0; 0, 0, 0)^t, \quad (0, 0, 0; 0, 1, 0; 0, 0, -1)^t,$$

and the kernel of $A[v_1]^\Gamma$ is spanned by

$$(0, 1, 0; -1, 0, 0; 0, 0, 0)^t,$$

$$(0, 0, 0; 0, 0, 1; 0, -1, 0)^t,$$

$$(0, 0, -1; 0, 0, 0; 1, 0, 0)^t.$$

If $x \otimes y \in \mathcal{RA}[v_1]$ and $\bar{x} \otimes y \in \mathcal{RA}[v_1]^\Gamma$, then \bar{x} and y are parallel to each other, and so we may assume that $\bar{x} = y$. Further, we have $|x_1| = |x_2| = |x_3|$. Therefore, we see that

$$(12) \quad Q[A[v_1]] = \{(\bar{x}_1, \bar{x}_2, \bar{x}_3)^t \otimes (x_1, x_2, x_3)^t : |x_1| = |x_2| = |x_3|\}.$$

Next, we turn our attention to $A[v_b]$ for $b \neq 1$. The kernel of $A[v_b]^\Gamma$ is spanned by

$$(0, b, 0; -1, 0, 0; 0, 0, 0)^t,$$

$$(0, 0, 0; 0, 0, b; 0, -1, 0)^t,$$

$$(0, 0, -1; 0, 0, 0; b, 0, 0)^t$$

If $\bar{x} \otimes y$ is a range vector of $A[v_b]^\Gamma$ then we have

$$(13) \quad b\bar{x}_1y_2 - \bar{x}_2y_1 = 0, \quad b\bar{x}_2y_3 - \bar{x}_3y_2 = 0, \quad b\bar{x}_3y_1 - \bar{x}_1y_3 = 0.$$

Multiplying the above equations, we have $b^3\bar{x}_1\bar{x}_2\bar{x}_3y_1y_2y_3 = \bar{x}_1\bar{x}_2\bar{x}_3y_1y_2y_3$, from which we see that at least one of x_i is zero. We also have $x_i = 0 \iff y_i = 0$ from (13). Therefore, we see that the set $Q[A[v_b]]$ consists of product vectors

$$(14) \quad \begin{aligned} & (0, \bar{x}_2, b\bar{x}_3)^t \otimes (0, x_2, x_3)^t, \\ & (b\bar{x}_1, 0, \bar{x}_3)^t \otimes (x_1, 0, x_3)^t, \\ & (\bar{x}_1, b\bar{x}_2, 0)^t \otimes (x_1, x_2, 0)^t. \end{aligned}$$

We proceed to show that $A[v_1]$ satisfies the condition (iv) of Theorem 2.2. We take arbitrary $z \in Q[A[v_1]]$, then we may assume that

$$z = (1, \bar{\alpha}, \bar{\beta})^t \otimes (1, \alpha, \beta)^t,$$

with $|\alpha| = |\beta| = 1$. Then we have the relation

$$A[v_1] = \frac{1}{9} \sum_{(\omega, \eta) \in \Omega \times \Omega} z_4(\alpha\omega, \beta\eta) z_4(\alpha\omega, \beta\eta)^*,$$

where $z_4(\cdot, \cdot)$ was defined in (10), and so we see that $A[v_1]$ satisfies the conditions in Theorem 2.2.

We note that the correspondence $x \mapsto A[x]$ is an injective affine map from the convex body C into the convex cone \mathbb{V}_1 , and the line segment from an interior point of the edge e_b to the vertex v_b cannot be extended within C whenever $b \neq 1$. This tells us that the

line segment from $A[e_b]$ to $A[v_b]$ cannot be extended within the cone \mathbb{V}_1 . We also note that $\mathcal{R}A[e_b] = \mathcal{R}A[v_b]$ and $\mathcal{R}A[e_b]^\Gamma = \mathcal{R}A[v_b]^\Gamma$, and so $\mathbb{T}[A[e_b]] = \mathbb{T}[A[v_b]]$. Therefore, the line segment from $A[e_b] \in \mathbb{V}_1 \cap \mathbb{T}[A[v_b]]$ to $A[v_b]$ cannot be extended within the face $\mathbb{V}_1 \cap \mathbb{T}[A[v_b]]$. This means that $A[v_b]$ is on the boundary point of the face $\mathbb{V}_1 \cap \mathbb{T}[A[v_b]]$, and so $A[v_b]$ does not satisfy the conditions in Theorem 2.2. We use the duality to overcome this difficulty. We see by a direct calculation

$$\langle A[v_1], \Phi(\frac{1}{b}) \rangle = \langle A[v_b], \Phi(\frac{1}{b}) \rangle = 0$$

that $A[v_1]$ and $A[v_b]$ belong to the dual face $\Phi(\frac{1}{b})'$ of the map $\Phi(\frac{1}{b}) \in \mathbb{P}_1$ with respect to dual pair $(\mathbb{V}_1, \mathbb{P}_1)$.

Now, we determine the set $\{z \in Q[A[v_b]] : zz^* \in \Phi(\frac{1}{b})'\}$. For $z = (\bar{x}_1, b\bar{x}_2, 0)^t \otimes (x_1, x_2, 0)^t$, we see that $zz^* \in \Phi(\frac{1}{b})'$ if and only if

$$\begin{aligned} |x_1|^4 \frac{(1-b)^2}{1-b+b^2} + |x_1 x_2|^2 \frac{b^2}{1-b+b^2} \\ + b^2 |x_1 x_2|^2 \frac{1}{1-b+b^2} + b^2 |x_2|^4 \frac{(1-b)^2}{1-b+b^2} - 2b |x_1 x_2|^2 = 0 \end{aligned}$$

if and only if

$$(|x_1|^2 - b|x_2|^2) = 0$$

if and only if the relation $|x_1|^2 = b|x_2|^2$ holds. By the similar way, we also see that the set $\{z \in Q[A[v_b]] : zz^* \in \Phi(\frac{1}{b})'\}$ consists of product vectors

$$\begin{aligned} (15) \quad & (0, \bar{x}_2, b\bar{x}_3)^t \otimes (0, x_2, x_3)^t \quad \text{with } |x_2|^2 = b|x_3|^2, \\ & (b\bar{x}_1, 0, \bar{x}_3)^t \otimes (x_1, 0, x_3)^t \quad \text{with } |x_3|^2 = b|x_1|^2, \\ & (\bar{x}_1, b\bar{x}_2, 0)^t \otimes (x_1, x_2, 0)^t \quad \text{with } |x_1|^2 = b|x_2|^2, \end{aligned}$$

as was done in [13]. We proceed to show that

$$\mathbb{V}_1[A[v_b]] = \mathbb{T}[A[v_b]] \cap \Phi(\frac{1}{b})'.$$

To do this, it suffices to show that $A[v_b]$ is an interior point of the convex set spanned by zz^* with z in (15). Take a z in (15), we may assume that z is one of the following forms

$$(0, 1, b\bar{\alpha})^t \otimes (0, 1, \alpha)^t, \quad (b\bar{\alpha}, 0, 1)^t \otimes (\alpha, 0, 1)^t, \quad (1, b\bar{\alpha}, 0)^t \otimes (1, \alpha, 0)^t,$$

with $b|\alpha|^2 = 1$. Note that we have

$$\begin{aligned} (0, 1, b\bar{\alpha})^t \otimes (0, 1, \alpha)^t &= \frac{1}{\sqrt{b}} z_1(\sqrt{b}\bar{\alpha}), \\ (b\bar{\alpha}, 0, 1)^t \otimes (\alpha, 0, 1)^t &= \frac{1}{\sqrt{b}} z_2(\sqrt{b}\bar{\alpha}), \\ (1, b\bar{\alpha}, 0)^t \otimes (1, \alpha, 0)^t &= \frac{1}{\sqrt{b}} z_3(\sqrt{b}\bar{\alpha}), \end{aligned}$$

where $z_i(\cdot)$ is defined as in the equation (9). Therefore, the above claim is proved by the relation

$$A[2, b, \frac{1}{b}] = \frac{1}{3b} \sum_{i=1}^3 \sum_{\omega \in \Omega} z_i(\sqrt{b}\bar{\alpha}\omega) z_i(\sqrt{b}\bar{\alpha}\omega)^*,$$

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where $\Omega = \{1, e^{\frac{2}{3}\pi i}, e^{-\frac{2}{3}\pi i}\}$ is again the third roots of unity.

Since $A[v_1]$ and $A[v_b]$ belong to the dual face $\Phi(\frac{1}{b})'$, we see that the convex hull generated by $A[v_1]$ and $A[v_b]$ is contained in the dual face $\Phi(\frac{1}{b})'$. For a product vector $z \in \mathbb{C}^3 \otimes \mathbb{C}^3$, we recall [13] that $zz^* \in \Phi(\frac{1}{b})'$ if and only if $z \in Q[A[v_1]]$ or z is of the form in (15). This means that the face $\Phi(\frac{1}{b})'$ is exactly the convex hull of two faces $\mathbb{V}_1[A[v_1]]$ and $\mathbb{V}_1[A[v_b]]$. Furthermore, since $A[v_1]$ and $A[v_b]$ are interior points of $\mathbb{V}_1[A[v_1]]$ and $\mathbb{V}_1[A[v_b]]$, respectively, and $A[e_b]$ is a nontrivial convex combination of $A[v_1]$ and $A[v_b]$, we conclude that $A[e_b]$ is an interior point of the convex hull of the faces $\mathbb{V}_1[A[v_1]]$ and $\mathbb{V}_1[A[v_b]]$. We summarize as follows:

Theorem 4.1. *Theorem We have the following:*

- (i) $\mathbb{V}_1[A[v_1]] = \mathbb{V}_1 \cap \mathbb{T}[A[v_1]]$.
- (ii) $\mathbb{V}_1[A[v_b]] = \Phi(\frac{1}{b})' \cap \mathbb{T}[A[v_b]]$, for every nonnegative real number b with $b \neq 1$.
- (iii) $\mathbb{V}_1[A[e_b]] = \Phi(\frac{1}{b})'$ is the convex hull of $\mathbb{V}_1[A[v_1]]$ and $\mathbb{V}_1[A[v_b]]$, for every nonnegative real number b with $b \neq 1$.

We note that both $\mathcal{RA}[e_b]$ and $\mathcal{RA}[e_b]^\Gamma$ are full spaces, and so $\mathbb{V}_1[A[e_b]]$ gives rise to an explicit example of proper face of \mathbb{V}_1 which is in the interior of the cone \mathbb{T} . Since $\Phi(\frac{1}{b})$ is exposed [13], we see that $\mathbb{V}_1[A[e_b]]$ is a maximal faces of \mathbb{V}_1 .

5. OTHER FACES

We have seen that $A[v_b]$ does not satisfy the conditions in Theorem 2.2 for $b \neq 1$. The exactly same argument shows that neither $A[v_0]$ nor $A[v_\infty]$ satisfy the conditions in Theorem 2.2, if we use (f, e_0) and (f, e_∞) in the places of (e_b, v_b) in the argument. In the remaining of this paper, we show that $A[e^1]$, $A[e^b]$ and $A[f]$ satisfy the conditions in Theorem 2.2. We will sustain the notation Ω for the set of third roots of unity.

First of all, we consider $A[e^b]$. Recall that $A[e^b]$ is an interior point of the face $\mathbb{T}[v_b]$. Therefore, $Q[e^b]$ coincides with $Q[v_b]$, and product vectors which belong to $Q[e^b]$ are of the forms in (14). In order to show that $A[e^b]$ satisfies the condition (iv) of Theorem 2.2, it suffices to consider the following product vector:

$$\begin{aligned}\tilde{z}_1(\alpha) &= (0, 1, b\bar{\alpha})^t \otimes (0, 1, \alpha)^t = (0, 0, 0; 0, 1, \alpha; 0, b\bar{\alpha}, b|\alpha|^2)^t, \\ \tilde{z}_2(\alpha) &= (b\bar{\alpha}, 0, 1)^t \otimes (\alpha, 0, 1)^t = (b|\alpha|^2, 0, b\bar{\alpha}; 0, 0, 0; \alpha, 0, 1)^t, \\ \tilde{z}_3(\alpha) &= (1, b\bar{\alpha}, 0)^t \otimes (1, \alpha, 0)^t = (1, \alpha, 0; b\bar{\alpha}, b|\alpha|^2, 0; 0, 0, 0)^t\end{aligned}$$

in $\mathbb{C}^3 \otimes \mathbb{C}^3$, for $\alpha \in \mathbb{C}$. We fix an interior point $(a, b, \frac{1}{b})$ in the interior of the edge e_b with $a > 2$. It is straightforward to see that

$$A[a, b, \frac{1}{b}] = A[2, b, \frac{1}{b}] + (a - 2) \sum_{i=1}^3 \tilde{z}_i(0) \tilde{z}_i(0)^*.$$

Now, we assume that $\alpha \neq 0$. Then we also have

$$A[a(\alpha), b, \frac{1}{b}] = \frac{1}{3b|\alpha|^2} \sum_{i=1}^3 \sum_{\omega \in \Omega} \tilde{z}_i(\alpha\omega) \tilde{z}_i(\alpha\omega)^*,$$

where

$$a(\alpha) = b|\alpha|^2 + \frac{1}{b|\alpha|^2} \geq 2.$$

If $a(\alpha) = a$, then we have done. If $a(\alpha) \neq a$, then we can choose $t_0 > 1$ such that

$$(1 - t_0)a(\alpha) + t_0a > 2.$$

Therefore, we can take a complex number β such that

$$a(\beta) = (1 - t_0)a(\alpha) + t_0a,$$

and we have the relation

$$A[a, b, \frac{1}{b}] = \frac{1}{t_0} A[a(\beta), b, \frac{1}{b}] + \left(1 - \frac{1}{t_0}\right) A[a(\alpha), b, \frac{1}{b}].$$

Consequently, we conclude that $A[e^b]$ satisfies the conditions in Theorem 2.2.

Next, we turn our attention to $A[e^1]$, that is, $A[a, 1, 1]$ for a fixed $a > 1$. First, it is easy to see that $Q[e^1]$ consists of product vectors

$$\tilde{z}(x_1, x_2, x_3) = (x_1, x_2, x_3)^t \otimes (\bar{x}_1, \bar{x}_2, \bar{x}_3)^t.$$

We take arbitrary $\tilde{z}(x_1, x_2, x_3)$ in $Q[e^1]$. If the only one x_i is nonzero in $\tilde{z}(x_1, x_2, x_3)$, then we have the relation

$$\begin{aligned} A[a, 1, 1] &= A[1, 1, 1] \\ &+ (a - 1) (\tilde{z}(1, 0, 0)\tilde{z}(1, 0, 0)^* + \tilde{z}(0, 1, 0)\tilde{z}(0, 1, 0)^* + \tilde{z}(0, 0, 1)\tilde{z}(0, 0, 1)^*). \end{aligned}$$

Now, we assume that at least two x_i 's are nonzero in $\tilde{z}(x_1, x_2, x_3)$, and define

$$\begin{aligned} k(x_1, x_2, x_3) &= |x_1|^2|x_2|^2 + |x_2|^2|x_3|^2 + |x_3|^2|x_1|^2, \\ a(x_1, x_2, x_3) &= \frac{|x_1|^4 + |x_2|^4 + |x_3|^4}{k(x_1, x_2, x_3)}. \end{aligned}$$

Then we get $a(x_1, x_2, x_3) \geq 1$, and it is straightforward to see that

$$\begin{aligned} &A[a(x_1, x_2, x_3), 1, 1] \\ &= \frac{1}{9k(x_1, x_2, x_3)} \sum_{(s_1, s_2, s_3) \in \Lambda} \sum_{(\omega, \eta) \in \Omega \times \Omega} \tilde{z}(s_1, \omega s_2, \eta s_3) \tilde{z}(s_1, \omega s_2, \eta s_3)^*, \end{aligned}$$

where $\Lambda = \{(x_1, x_2, x_3), (x_2, x_3, x_1), (x_3, x_1, x_2)\}$. If $a(x_1, x_2, x_3) = a$, then we have done. If $a(x_1, x_2, x_3) \neq a$, then we can choose $t_0 > 1$ and \tilde{x}_1 such that

$$a(\tilde{x}_1, 1, 1) = (1 - t_0)a(x_1, x_2, x_3) + t_0a > 1,$$

and so we have the relation

$$A[a, 1, 1] = \frac{1}{t_0} A[a(\tilde{x}_1, 1, 1), 1, 1] + \left(1 - \frac{1}{t_0}\right) A[a(x_1, x_2, x_3), 1, 1].$$

Consequently, we see that $A[e^1]$ satisfies the conditions of Theorem 2.2.

Finally, we show that $A[f]$ satisfies the condition (iv) of Theorem 2.2. We begin with description of $Q[A[f]]$. It is easy to see that

$$Q[A[f]] = \{(x_1, x_2, x_3)^t \otimes (y_1, y_2, y_3)^t : x_1 y_1 = x_2 y_2 = x_3 y_3\}.$$

Therefore, we may assume that a product vector in $Q[A[f]]$ is one of the following vectors:

$$\begin{aligned}
(16) \quad & v_1(s, t) = (1, 0, 0)^t \otimes (0, s, t)^t = (0, s, t, ; 0, 0, 0, ; 0, 0, 0)^t, \\
& v_2(s, t) = (0, 1, 0)^t \otimes (s, 0, t)^t = (0, 0, 0, ; s, 0, t, ; 0, 0, 0)^t, \\
& v_3(s, t) = (0, 0, 1)^t \otimes (s, t, 0)^t = (0, 0, 0, ; 0, 0, 0, ; s, t, 0)^t, \\
& v_4(s, t) = (0, s, t)^t \otimes (1, 0, 0)^t = (0, 0, 0, ; s, 0, 0, ; t, 0, 0)^t, \\
& v_5(s, t) = (s, 0, t)^t \otimes (0, 1, 0)^t = (0, s, 0, ; 0, 0, 0, ; 0, t, 0)^t, \\
& v_6(s, t) = (s, t, 0)^t \otimes (0, 0, 1)^t = (0, 0, s, ; 0, 0, t, ; 0, 0, 0)^t, \\
& v_7(x_2, x_3) = (1, x_2, x_3)^t \otimes (1, \frac{1}{x_2}, \frac{1}{x_3})^t = (1, \frac{1}{x_2}, \frac{1}{x_3}, ; x_2, 1, \frac{x_2}{x_3}, ; x_3, \frac{x_3}{x_2}, 1)^t,
\end{aligned}$$

where s and t are arbitrary complex numbers, and x_2 and x_3 are nonzero complex numbers. Note that for $i = 1, 2, \dots, 7$ and for any given pair (s_i, t_i) of complex numbers with $s_7, t_7 \neq 0$, we can find pairs of complex numbers (s_j, t_j) ($1 \leq j \neq i \leq 7$) satisfying the following two conditions

$$(17) \quad \frac{1}{|s_7|^2} + |s_1|^2 + |s_5|^2 = \frac{|s_7|^2}{|t_7|^2} + |t_2|^2 + |t_6|^2 = |t_7|^2 + |s_3|^2 + |t_4|^2,$$

and

$$(18) \quad \frac{1}{|t_7|^2} + |s_6|^2 + |t_1|^2 = \frac{|t_7|^2}{|s_7|^2} + |t_3|^2 + |t_5|^2 = |s_7|^2 + |s_2|^2 + |s_4|^2.$$

For a 7-tuple $\mathcal{V} = (v_1(s_1, t_1), v_2(s_2, t_2), \dots, v_7(s_7, t_7))$ of the product vectors satisfying the conditions (17) and (18), one can easily show that

$$(19) \quad \frac{1}{9} \sum_{i=1}^7 \sum_{(\omega, \eta) \in \Omega \times \Omega} v_i(\omega s_i, \eta t_i) v_i(\omega s_i, \eta t_i)^* = A[1, b(\mathcal{V}), c(\mathcal{V})]$$

where $b(\mathcal{V})$ denotes the value in (18), and $c(\mathcal{V})$ denotes the value in (17). From the following inequalities

$$\begin{aligned}
& \left(\frac{1}{|t_7|^2} + |s_6|^2 + |t_1|^2 \right) (|t_7|^2 + |s_3|^2 + |t_4|^2) \geq 1, \\
& \left(\frac{1}{|s_7|^2} + |s_1|^2 + |s_5|^2 \right) (|s_7|^2 + |s_2|^2 + |s_4|^2) \geq 1, \\
& \left(\frac{|s_7|^2}{|t_7|^2} + |t_2|^2 + |t_6|^2 \right) \left(\frac{|t_7|^2}{|s_7|^2} + |t_3|^2 + |t_5|^2 \right) \geq 1,
\end{aligned}$$

we know that $b(\mathcal{V})c(\mathcal{V}) \geq 1$. Furthermore, we can show that

$$(20) \quad b(\mathcal{V}) \geq 1, \quad c(\mathcal{V}) \geq 1.$$

To see this, we assume that $b(\mathcal{V}) < 1$ in (18). Then it gives rise to the following inequalities:

$$(21) \quad \frac{1}{|t_7|^2} < 1, \quad \frac{|t_7|^2}{|s_7|^2} < 1, \quad |s_7|^2 < 1.$$

From the first two inequalities in (21), we obtain $\frac{1}{|s_7|^2} < 1$. This contradicts the inequality $|s_7|^2 < 1$ in (21). Therefore, we conclude $b(\mathcal{V}) \geq 1$. Similarly, we can show that $c(\mathcal{V}) \geq 1$.

For the equality in (20), we have $b(\mathcal{V}) = c(\mathcal{V}) = 1$ holds if and only if the only nonzero product vector in 7-tuple \mathcal{V} is $v_7(s_7, t_7)$ with $|s_7| = |t_7| = 1$. In this case, the relation (19) is reduced to the relation (11). Note that $A[v_1]$ is on the boundary of the face $\mathbb{T}[A[f]]$.

Now we proceed to show that $A[f]$ satisfies the condition (iv) of Theorem 2.2. We take arbitrary $z \in Q[A[1, b, c]]$ for fixed $b > 1$ and $c > 1$. Then we may assume that z is one of the vectors in (16). For this product vector, we can find 7-tuple \mathcal{V} satisfying the condition (19). Furthermore, we can choose $t_0 > 1$ satisfying the conditions

$$(1 - t_0)b(\mathcal{V}) + t_0b > 1 \text{ and } (1 - t_0)c(\mathcal{V}) + t_0c > 1.$$

Then we take vectors $v_1(\tilde{s}_1, \tilde{t}_1)$, $v_2(\tilde{s}_2, \tilde{t}_2)$ and $v_3(\tilde{s}_3, \tilde{t}_3)$ in (16) as follows:

$$\begin{aligned}\tilde{t}_1 &= \tilde{s}_2 = \tilde{t}_3 = (1 - t_0)b(\mathcal{V}) + t_0b - 1, \\ \tilde{s}_1 &= \tilde{t}_2 = \tilde{s}_3 = (1 - t_0)c(\mathcal{V}) + t_0c - 1.\end{aligned}$$

Consequently, we have the relation

$$A[1, b, c] = \frac{1}{t_0} \left(A[1, 1, 1] + \sum_{i=1}^3 v_i(\tilde{s}_i, \tilde{t}_i) v_i(\tilde{s}_i, \tilde{t}_i)^* \right) + \left(1 - \frac{1}{t_0} \right) A[1, b(\mathcal{V}), c(\mathcal{V})].$$

This shows that $A[f]$ satisfies the conditions of Theorem 2.2. We summarize as follows:

Theorem 5.1. *Theorem We have the following:*

- (i) $\mathbb{V}_1[A[e^1]] = \mathbb{V}_1 \cap \mathbb{T}[A[e^1]] = \Phi(1)'$.
- (ii) $\mathbb{V}_1[A[e^b]] = \mathbb{V}_1 \cap \mathbb{T}[A[e^b]]$, for every nonnegative real number b with $b \neq 1$.
- (iii) $\mathbb{V}_1[A[f]] = \mathbb{V}_1 \cap \mathbb{T}[A[f]]$.

Proof. It remains that $\mathbb{V}_1 \cap \mathbb{T}[A[e^1]] = \Phi(1)'$. It is straightforward to show that $zz^* \in \Phi(1)'$ for all product vector $z \in Q[A[e^1]]$. So we have the inclusion

$$\mathbb{V}_1 \cap \mathbb{T}[A[e^1]] \subset \Phi(1)'.$$

For any product vector $z = (x_1, x_2, x_3)^t \otimes (y_1, y_2, y_3)^t$, we see that

$$\begin{aligned}zz^* \in \Phi(1)' &\iff |x_1\bar{y}_2 - x_2\bar{y}_1|^2 + |x_1\bar{y}_3 - x_3\bar{y}_1|^2 + |x_2\bar{y}_3 - x_3\bar{y}_2|^2 = 0 \\ &\iff (\bar{x}_1, \bar{x}_2, \bar{x}_3)^t \text{ is parallel to } (y_1, y_2, y_3)^t \\ &\iff z \in Q[A[e^1]].\end{aligned}$$

Therefore we can conclude that $\Phi(1)'$ coincides with the face $\mathbb{V}_1 \cap \mathbb{T}[A[e^1]]$. \square

In conclusion, we exhibited examples of separable states which are on the boundary of the convex cone \mathbb{V}_1 generated by all separable states, but lie in the interior of the convex cone \mathbb{T} of all PPT states. We also showed that they determine maximal faces of the cone \mathbb{V}_1 whose interiors lie in the interior of the cone \mathbb{T} , and found all extreme rays in these maximal faces. We note that there are recent remarkable progresses [1, 2] to understand the facial structures of the convex cone \mathbb{V}_1 . They are very useful when we consider the faces of the cone \mathbb{V}_1 which is determined by separable states whose range dimensions are low. It is the hope of the authors that our examples motivate general framework to deal with faces of the cone \mathbb{V}_1 determined by arbitrary separable states.

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FACULTY OF MATHEMATICS AND STATISTICS, SEJONG UNIVERSITY, SEOUL 143-747, KOREA

DEPARTMENT OF MATHEMATICS AND INSTITUTE OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY,
SEOUL 151-742, KOREA